THE UNIVERSITY OF AKRON Mathematics and Computer Science

Lesson 7: Solving Equations & Inequalities

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7. Solving Equations & Inequalities

There are two basic tools for solving equations: (1) Adding the same expression to both sides of an equation and (2) multiplying both sides of the equation by the same expression. In symbols,

$$a = b \iff a + c = b + c \tag{1}$$

$$a = b \iff ac = bc \qquad c \neq 0$$
 (2)

Here the symbol \iff means "if and only if" which is a fancy way of saying "is equivalent to." Equation (2) has an obvious variation

$$a = b \iff \frac{a}{c} = \frac{b}{c} \qquad c \neq 0$$
 (3)

Some care needs to be taken when applying (2) and (3) in the case where c is an algebraic expression containing unknowns. Usually, students have no problems when c is a numerical value.

7.1. How to Solve Linear Equations

A linear equation is an equation of the form

$$ax + b = 0$$
 $a \neq 0$

The solution set would represent the zeros or roots of the linear polynomial ax + b.

Even though the equation is very simple, student still make errors solving it. Here are some representative examples. All linear equations are solved in the same way.

EXAMPLE 7.1. Solve each linear equation. (a) 4x + 5 = 0 (b) $\frac{1}{2}x - 4 = 6$ (c) 7 - 3x = 2

Strategy for Solving. Isolate the unknown, x say, on one side of the equation with other terms of the equation on the other side. Divide through both sides by the coefficient of the unknown to obtain the solution.

EXERCISE 7.1. Solve for x in each of the following using good techniques (as exhibited in EXAMPLE 7.1). Passing is 100% correct.

(a)
$$4x - 8 = 0$$
 (b) $23 - 2x = 3$ (c) $\frac{4}{3}x - 2 = 3$ (d) $\frac{2}{3}x + \frac{4}{5} = \frac{8}{3}$

Here is a *slight variation* on the type of linear equation previously considered.

EXERCISE 7.2. Solve each for x. (a) 3x - 2 = 7x + 3 (b) 9x + 3 = 4x - 2 (c) $\frac{1}{2}x = \frac{4}{3}(x - 3)$

Sometimes we have equations that have several symbolic quantities. Examine each of the following problems.

EXERCISE 7.3. In each of the equations listed below, solve for the indicated variable. Solve for ...

(a) x in 5x - 3y = 4 (b) y in 5x - 3y = 4(c) $z \text{ in } x^2z - 12x + y = 1$

It is apparent from the above examples and exercises that the operations of adding (1), multiplying (2), and dividing (3) both sides of an equation by the same expression are standard and useful tools in your

toolbox of techniques for solving equations. Do not make up your own methods, use the standard ones and ... these are they!

When working out mathematical problems it is important to organize your thoughts on paper properly and clearly. After solving a problem, recopy it neatly. Copy the style of this tutorial or some other textbook. Try to improve your handwriting. Use proper notation.

• Essentially Linear Equations

Some equations are 'disguised' linear equations. They do not require any special techniques other than what is needed to solve linear equations.

EXAMPLE 7.2. Solve for x in each of the following.

(a)
$$(x-1)^2 = x^2$$
 (b) $\frac{2x^2 + 5x - 1}{x+1} = 2x+1$
(c) $\frac{1-3x}{2x+1} = 4$ (d) $\frac{4}{3x+1} = 1$

Now, consider the following set of exercises.

EXERCISE 7.4. Solve for x is each of the following. (a) $\frac{x}{3x+8} = 5$ (b) $\frac{5x+2}{3-8x} = 2$ (c) $(2x-3)^2 = (2x-7)^2$ (d) $\frac{8x^2+2x+1}{2x+1} = 4x+1$

7.2. Solving Second Degree Equations

We now turn to the problem of solving equations of the form:

$$ax^2 + bx + c = 0. (4)$$

There are two standard methods of solving this kind of equation: (1) by factoring the left-hand side and (2) by applying the Binomial Formula.

• Factoring Methods

We have already studied techniques of factoring polynomials of degree two; therefore, it is not necessary to look at a large number of examples. If necessary, review factoring.

The method of factoring can certainly be applied to any polynomial equation and is not restricted to quadratic equations. In addition to factoring, a major tool used in solving equations is the ZERO-PRODUCT PRINCIPLE.

ZERO-PRODUCT PRINCIPLE

$$ab = 0 \implies a = 0 \text{ or } b = 0$$
 (5)

This principle states the obvious property of the real number system: The only way the product of two numbers can be zero is if one of them is zero.

The following example illustrates standard reasoning and solution methods. Read carefully.

EXAMPLE 7.3. Solve each of the following. (a) $x^2 - 5x + 6 = 0$ (b) $x^2 + 4x + 4 = 0$ (c) $6x^2 - x - 2 = 0$

Now using the same methods as exhibited in the previous example solve each of the following. Passing is 100%.

EXERCISE 7.5. (Skill Level 1) Solve for x in each of the following.

(a)
$$x^2 - 7x + 12 = 0$$
 (b) $x^2 + 3x = 10$ (c) $\frac{x^2 + 5x - 6}{x^2 + 1} = 0$

Here are a few more quadratic equations.

EXERCISE 7.6. Solve for x in each of the following.

(a)
$$12x^2 - 17x + 6 = 0$$
 (b) $20x^2 + 3x = 2$ (c) $\frac{4x^2 - 4x + 1}{x - 1} = 0$

Factorization techniques are not limited to second degree equations. Here are a few higher degree equations that can be factored fairly

easily. Some of the fourth degree equations below can be solved using the factorization techniques for quadratic polynomials.

EXERCISE 7.7. Solve for x in each of the following.
(a)
$$x^3 - 2x^2 - 3x = 0$$
 (b) $x^4 - 16 = 0$
(c) $x^4 - 2x^2 - 3 = 0$ (d) $x^4 - 5x^2 + 6 = 0$

• Completing the Square

In addition to factorization methods, the technique of **completing the square** can also be used to solve a quadratic equation. Even though this technique will be used to solve equation, completion of the square has certain uses in other kinds of mathematical problems. If you go on to **Calculus**, for example, you will see it within the context of integration problem.

Completion of the Square Algorithm. Below are the steps for completing the square with an abstract and a particular equation that illustrate the steps.

1. Problem: Solve the quadratic equation for x:

$$ax^2 + bx + c = 0$$
 $2x^2 + 12x - 3 = 0.$

2. Associate the x^2 term and the x term:

$$(ax^{2} + bx) + c = 0$$
 $(2x^{2} + 12x) - 3 = 0$

3. Factor out of the coefficient of the x^2 term.

$$a(x^{2} + \frac{b}{a}x) + c = 0$$
 $2(x^{2} + 6x) - 3 = 0$

4. Take one-half of the *coefficient* of the x and *square it*:

5. Take this number, and add it *inside the parentheses*; this addition must be compensated for by *subtracting it from outside the parentheses*:

$$a(x^{2} + \frac{b}{a}x + \frac{b^{2}}{4a^{2}}) + c - \frac{b^{2}}{4a} = 0 \qquad 2(x^{2} + 6x + 9) - 3 - 18 = 0$$

Care must be made here because we are adding the term *in-side* the parentheses and subtracting an *equal* quantity *outside* the parentheses. Study the abstract version and the particular example closely to understand what is meant.

6. The trinomial inside the parentheses is a perfect square:

$$a(x+\frac{b}{2a})^2 + c - \frac{b^2}{4a} = 0 \qquad 2(x+3)^2 - 21 = 0 \qquad (6)$$

End Complete Square

What does this accomplish? Observe that the equations in (6) can be rewritten as

$$AX^2 = C \tag{7}$$

where X is a linear polynomial. This kind of equation can easily be solved as follows:

$$AX^2 = C \implies X^2 = \frac{C}{A} \implies X = \pm \sqrt{\frac{C}{A}}.$$

This sequence of steps can be carried out in every case.

Let's illustrate by continuing to solve the equation carried in the completion of the square algorithm.

EXAMPLE 7.4. Solve the equation $2x^2 + 12x - 3 = 0$.

EXAMPLE 7.5. Solve by completing the square: $3x^2 + 2x - 5 = 0$.

EXERCISE 7.8. Solve each of the following by completing the square. (a) $8x^2 - 2x - 1 = 0$ (b) $3x^2 + 5x + 2 = 0$ (c) $x^2 + x - 1 = 0$

All the above examples and exercises were done *exacly* the same way. The method of completion of the square is a useful tool when solving quadratic equations, but it is not the most efficient method. The next section on the QUADRATIC FORMULA is a more standard tool than is completing the square.

Despite its inefficiencies, completion of squares is still is useful technique to know as it is use elsewhere in mathematics.

• The Quadratic Formula

The solutions of equation (4) can be found in a more direct way than the method of factorization or completing the square by using the so-called QUADRATIC FORMULA. Let's state/prove this formula.

Theorem. Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad a \neq 0 \tag{8}$$

(1) If
$$b^2 - 4ac < 0$$
, (8) has no solutions;
(2) if $b^2 - 4ac = 0$, (8) has only one solution;
(3) if $b^2 - 4ac > 0$, (8) has two distinct solutions.

In the latter two cases, the solutions are given by the QUADRATIC FORMULA:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9}$$

Proof.

Theorem Notes: The expression b^2-4ac is called the **discriminant** for the quadratic equation. It can be used, at casual glance, to determine whether a given equation has one, two, or no solutions.

• The discriminant is a handy way of classifying a polynomial $P(x) = ax^2 + bx + c$ as irreducible or not. The polynomial P(x) is irreducible if and only if its discriminant is negative: $b^2 - 4ac < 0$.

Quiz. Using the discriminant, $b^2 - 4ac$, respond to each of the following questions.

1. Is the quadratic polynomial $x^2 - 4x + 3$ irreducible?

(a) Yes (b) No

2. Is the quadratic polynomial $2x^2 - 4x + 3$ irreducible?

(a) Yes (b) No

3. How many solutions does the equation $2x^2 - 3x - 2 = 0$ have? (a) none (b) one (c) two EndQuiz.

Let's go to the examples.

EXAMPLE 7.6. Solve each of the following using the QUADRATIC FORMULA

(a) $x^2 - 5x + 6 = 0$ (b) $x^2 + 4x + 4 = 0$ (c) $6x^2 - x - 2 = 0$ (d) $3x^2 - 3x + 1 = 0$

EXERCISE 7.9. Using the quadratic formula, solve each of the following.

(a) $2x^2 + 5x - 12 = 0$ (b) $3x^2 - 7x + 1 = 0$ (c) $x^2 + 1 = 0$ (d) $x^2 + x = 3$

Recognition. One problem students have is recognizing a quadratic equation. This is especially true when the equation has several symbolic expressions in it. Basically, a **quadratic equation in** x is an equation in which x^2 appears as a **term** and x appears as a **term**. The coefficient of the x^2 term is the value of a; the coefficient of the x term is the value of b; and all other terms comprise the value of c. The symbol x may be some other letter like y or z, but it can also be a compound symbol like x^2 , y^3 , or even something like $\sin(x)!$

Illustration 1. Here are some examples of quadratic equations.

- (a) The equation $3zx^2 sx + 4 = z$ is quadratic in x: a = 3z, b = -sand c = 4 - z. (Recall: We must write the equation in the form $3zx^2 - sx + (4 - z) = 0$).)
- (b) The equation $5wx^4 2w^2x^2 + 3 = 0$ is quadratic in x^2 and also quadratic in w.
- (c) The equation $4\sin^2(x) 4\sin(x) + 1 = 0$ is a quadratic equation in the $\sin(x)$.

Comments: In example (b), the fact that the equation is quadratic in x^2 means we can use the quadratic formula to solve for x^2 . The same equation is quadratic in w means that we can use the quadratic formula to solve for w. That's the way it works!

Part of the power of the QUADRATIC FORMULA is that it is a very efficient way of solving quadratic equations—more efficient than factoring, usually—and it can be applied even when the coefficients a, b, and c are symbolic. Here is a simple example.

Illustration 2. Solve for x in the equation $yx^2 - 2x - 4y^2 = 0$. This is a quadratic equation in x because the highest power of x is power 2. We can apply the QUADRATIC FORMULA with a = y, b = -2 and $c = -4y^2$:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(y)(-4y^2)}}{2y}$$
$$= \frac{2 \pm \sqrt{4 + 16y^3}}{2y} = \frac{2 \pm 2\sqrt{1 + 4y^3}}{2y}$$

(In the last step, I've extracted 4 from the radical, which appears as a 2 outside the radical.)

Note that we only get a solution when $1 + 4y^3 \ge 0$. This occurs when $y \ge -1/\sqrt[3]{4}$. (Solving Inequalities is taken up later.)

Thus, for any $y, y \ge -1/\sqrt[3]{4}$, the solutions for x are

$$x = \frac{1 + \sqrt{1 + 4y^3}}{y}, \quad x = \frac{1 - \sqrt{1 + 4y^3}}{y}$$

EXERCISE 7.10. The equation that appears in ILLUSTRATION 2 is a quadratic equation in y. Use the QUADRATIC FORMULA to solve for y.

EXERCISE 7.11. Use the QUADRATIC FORMULA to (a) solve for w in ILLUSTRATION 1; and (b) solve for x^2 in the same equation.

7.3. Solving Inequalities

We encounter the problem of solving inequalities in a variety of settings. For example, we stumbled across questions of solving inequalities in the previous section on the Quadratic Formula. In that section, our solutions came out in terms of symbolic quantities; recall the solution to the equation discussed in ILLUSTRATION 2:

$$x = \frac{1 \pm \sqrt{1 + 4y^3}}{y}$$

We only have "real solutions" (as opposed to solutions that are complex numbers) provided

$$1+4y^3 \ge 0.$$

This is a (rather simple) problem of solving an inequality.

• Tools for Solving Inequalities

When you manipulate an inequality for the purpose of trying to isolate the unknown on one side of the inequality, there are a few things you should know.



Tool Notes: Equation (10) states that you can add the same quantity to both sides of an inequality, and the inequality is preserved.

• By equation (11), you can multiply both sides of an inequality by a *positive number* and the inequality will be preserved.

• An important variation on this is equation (12): If you multiply both sides by a *negative number* the inequality is *reversed*. Caveat. It is property (12) that most students have trouble with.

• Finally, we can take a root of both sides and the inequality is preserved. There is a built-in proviso: Provided the n^{th} roots of both sides exist. This is not a problem when n is odd, but becomes one when n is even.

EXAMPLE 7.7. Let's finish the analysis of ILLUSTRATION 2: In the solution

$$x = \frac{1 \pm \sqrt{1 + 4y^3}}{y}$$

we require $1 + 4y^3 \ge 0$. Solve for y in this inequality.

• Simple Inequalities

The basic tools can be used to solve a simple class of inequalities: Essentially Linear Inequalities.

EXAMPLE 7.8. Solve each of the following inequalities for x. (a) 5x + 7 < 0 (b) $3 - 9x \ge 4$ (c) $3x^5 + 4 \ge 9$ (d) $3x^2 + 4 \le 3$

EXERCISE 7.12. Solve each of the following inequalities for x. Write your solutions in *interval notation*.

(a) $4x + 12 \le 5$ (b) $\frac{1}{2}x - \frac{3}{2} > 4$ (c) $3 - 8x \ge 4$

Here's a slight variation on the previous problems.

EXERCISE 7.13. Solve each inequality for x. Leave your answer in *set* notation.

(a) $2x - 1 \le 5x + 2$ (b) 3x + 4 > 1 - 6x

• Double Inequalities

By double inequalities I mean inequalities of the form

$$a \le b \le c \tag{14}$$

as well as all variations on same. This notation is short-hand for

$$a \leq b$$
 and $b \leq c$.

In this section we look at one simple type of problem—the case where the b in (14) is a linear polynomial and a and b are constants.

This kind of problem can be solve in much the same way as in the previous paragraphs.

Illustration 3. Solve for x in $-4 \le 3x - 1 < 6$. Solution:

$$\begin{array}{r|l} -4 \leq 3x-1 < 6 & \triangleleft \text{ given} \\ -4+1 \leq 3x < 6+1 & \triangleleft \text{ add 1 to all sides} \\ -3 \leq 3x < 7 & \triangleleft \text{ combine} \\ -1 \leq x < \frac{7}{3} & \triangleleft \text{ multiply all sides by} \end{array}$$

Presentation of Solution. Set Notation:

$$\{ x \mid -1 \le x < \frac{7}{3} \}$$

1/3

Interval Notation:

$$\left[-1, \frac{7}{3}\right)$$

Illustration Notes: The standard tools for manipulating inequalities hold. You can add the same quantity to *all* sides of the inequalities and you can multiply the same quantity to *all* sides.

• If you multiply all sides by a *negative number*, *all* the inequalities are reversed!

EXERCISE 7.14. Solve each of the following for x. Use interval notation to express your answer. Passing is 100%.

(a) $3 < 2x + 1 \le 12$ (b) $-2 \le 2 - 5x \le -1$ (c) $1 < \frac{3}{2}x + 1 < 4$

Quiz.

- 1. What is the solution set to the inequality $1 \le x \ge -1$? (a) $[1, +\infty)$ (b) [-1, 1] (c) \emptyset (i.e., no solution)
- 2. What is the solution to the inequality $-1 \ge x \ge 1$? (a) [-1,1] (b) $[1,+\infty)$ (c) \emptyset (i.e., no solution)
- 3. What is the solution to the inequality 3 ≤ 2x 1 ≤ 1?
 (a) [1,2]
 (b) Ø (i.e., no solution)
 EndQuiz.

• The Method of Sign Charts

More complex inequalities require different methods. The **Method of Sign Charts** is a general method of analyzing inequalities provided you can *factor* the algebraic expressions. Let me illustrate this method with an example.

Suppose you have an inequality of the form

 $R(x) \ge 0$

where R(x) is a rational expression. The idea is to factor the expression R(x) completely—both numerator and denominator—and analyze the sign of each factor. The **Sign Chart** is just a graphical way of storing all the information.

The next example is important because it delineates the steps and the reasoning that goes into the Sign Chart Method. **Read this example carefully.**

EXAMPLE 7.9. Solve the inequalities for x: (a) $x^2 - 3x + 2 \ge 0$ (b) $x \le x^2$.

EXERCISE 7.15. Solve for x in each of the following. (a) $x^2 - x - 2 \le 0$ (b) $x^3 - 4x^2 + 3x > 0$

Tip. The basis for the **Sign Chart Method** is a comparison of some algebraic expression with zero:

R(x) < 0 or R(x) > 0 or $R(x) \le 0$ or $R(x) \ge 0$

Therefore, the first step is to put your inequality in one of the above forms.

EXERCISE 7.16. Solve each of the following for x. Write your answers in *interval notation*.

(a)
$$x^2 - 3x \le 4$$
 (b) $6x - x^2 < 5$

This method is not limited to polynomial inequalities. The next example illustrates the method for an inequality involving a rational expression.

EXAMPLE 7.10. Solve the inequality for
$$x: \frac{x^2 - 4}{x - 3} < 0$$
.

EXERCISE 7.17. Solve for x in each of the following. (a) $\frac{x^2 - 2x}{x+1} < 0$ (b) $\frac{x}{x^2 - 3x + 2} \ge 0$

The *Sign Chart Method* is quite general and can be applied in a wide variety of problems. To illustrate this statement, the next exercise continues EXERCISE 7.10, which, in turn, was problem originally started in ILLUSTRATION 2.

EXERCISE 7.18. In order to have real solutions to the equation in EXERCISE 7.10, we require that the radicand in the solution

$$y = \frac{x^2 \pm \sqrt{x^4 - 32x}}{8}$$

be nonnegative. Find all x for which $x^4 - 32x \ge 0$.

Let's finish this section with a more challenging problem.

• EXERCISE 7.19. From EXERCISE 7.11, we used the quadratic formula to solve a fourth degree equation in x having symbolic coefficients in terms of w. We obtained

$$x^2 = \frac{w^2 \pm \sqrt{w^4 - 15w}}{5w} \tag{15}$$

In order for there to be real solutions (solutions belonging to the real number system), we require (1) $w^4 - 15w \ge 0$ and (2) the right-hand side of (15) to be nonnegative. Your assignment, should you decide to accept it, find all values of w that satisfy conditions (1) and (2).

7.4. Solving Absolute Inequalities

We finish this lesson by a short study of inequalities that involve the absolute value function. We look at two type of inequalities:

$$|a| < b$$
 and $|a| > b$.

These two are a fundamental type in inequality that occurs rather frequently in **Calculus**.

The method of solution is to (1) remove the absolute value ... in a legal way; (2) solve the resultant inequality using standard techniques described previously.

• Solving the Inequality |a| < b

The key to removing the absolute value from |a| < b is in the interpretation of absolute value. Recall that

|a| = the distance *a* is from zero (0).

The inequality |a| < b then states that a is less than b units from the origin. The fact that a is less than b units away from the origin would be equivalent to saying that a is between -b and b. That is,

|a| < b is equivalent to -b < a < b

Let's elevate this observation to the status of shadow box.

How to remove the
$$|\cdot|$$
 from $|a| < b$.
 $|a| < b \iff -b < a < b$ (16)

Note. Naturally, a similar statement is true for $|a| \leq b$; this is true if and only if $-b \leq a \leq b$.

Once the absolute value is removed, we now have a double inequality—which we solve.

EXAMPLE 7.11. Solve each of the following for x. (a) |x - 4| < 3 (b) |3x - 1| < 2 (c) $|2 - 4x| \le 5$

The method is simple enough, remove the absolute value, then solve the double linear inequality.

EXERCISE 7.20. Solve each of the following for x. Leave your answer in *interval notation*. Passing is 100%.

(a)
$$|x+3| < 8$$
 (b) $|4x+9| \le 1$ (c) $|2-7x| \le 3$

• Solving the Inequality |a| > b

The solution to the inequality |a| > b can be concluded once the absolute value as been removed.

A number a satisfies the inequality |a| > b if the distance it is away from the origin is greater than b units. This means the value of a is not in the interval [-b, b] (for these are exactly the numbers that are within b units of the origin). The number a we are looking for is not in the interval [-b, b]; therefore, the number a we seek must be greater than b or less than -b. In symbols ...

How to remove the
$$|\cdot|$$
 from $|a| > b$.
 $|a| > b \iff a > b$ or $a < -b$ (17)

Note. A similar statement is true for $|a| \ge b$: $|a| \ge b$ if and only if $a \ge b$ or $a \le -b$.

The Split-Solve-Join Solution Method. The process of solving absolute inequalities requires three steps.

- 1. Split: Split the absolute inequality using display line (17),
- 2. Solve: Solve the two inequalities, obtaining their solution sets.
- 3. Join: Join (take the union of) the solution sets from *Step* 2 to obtain the final solution set to your absolute inequality.

Here are a couple of examples that illustrate the SSJS Method. Read carefully.

EXAMPLE 7.12. Solve each of the following for x.

(a)
$$|x-3| > 4$$
 (b) $|5x+1| \ge 3$

EXERCISE 7.21. Using the SSJS Method, solve each of the absolute inequalities for x. Write your answer in *interval notation*.

(a)
$$|9x - 2| \ge 3$$
 (b) $|2 - 3x| > 6$ (c) $|\frac{3}{2}x + 2| > \frac{1}{3}$

These same methods can be applied to more complicated absolute inequalities. Here we have just considered *absolute linear inequalities*.

Equations and inequalities are the way we can pose questions; therefore, it is essential that we have very solid methods of solving equations and inequalities. The techniques are demonstrated in this lesson are far from being comprehensive, yet they will suffice you as you continue to learn.

In the next lesson, we introduce the **Cartesian Coordinate System** and discuss **Functions**. In LESSON 9 you will see how we use equations and inequalities to pose questions; the techniques of this lesson are then applied to solve.

Click here to continue to Lesson 8.

Solutions to Exercises

7.1. *Answers*:

(a)
$$4x - 8 = 0 \implies 4x = 8 \implies \boxed{x = 2}$$

(b) $23 - 2x = 3 \implies -2x = -20 \implies \boxed{x = 10}$
(c) $\frac{4}{3}x - 2 = 3 \implies \frac{4}{3}x = 5 \implies x = \frac{3}{4}(5) \implies \boxed{x = \frac{15}{4}}$

Part (d) on the next page.

Solutions to Exercises (continued)

(d)
$$\frac{2}{3}x + \frac{4}{5} = \frac{8}{3}$$
.
Solution: The most difficult thing about this one is the arithmetic!

$$\frac{2}{3}x + \frac{4}{5} = \frac{8}{3} \qquad \triangleleft \text{ given}$$

$$\frac{2}{3}x = \frac{8}{3} - \frac{4}{5} \qquad \triangleleft \text{ subtract } 4/5 \text{ both sides}$$

$$\frac{2}{3}x = \frac{28}{15} \qquad \triangleleft \text{ arithmetic}$$

$$x = \frac{3}{2} \cdot \frac{28}{15} \qquad \triangleleft \text{ multiply both sides by } 3/2$$

$$x = \frac{14}{5}$$

Presentation of Answer:
$$x = \frac{14}{5}$$

Exercise 7.1. \blacksquare

Solutions to Exercises (continued)

7.2. *Solutions*:

(a) Solve for
$$x: 3x - 2 = 7x + 3$$
.
 $3x - 2 = 7x + 3 \triangleleft \text{given}$
 $3x = 7x + 5 \triangleleft \text{add } 2 \text{ to both sides}$
 $-4x = 5 \triangleleft \text{add } -7x \text{ to both sides}$
 $x = -\frac{5}{4} \triangleleft \text{divide by } -4$
Presentation of Answer: $x = -\frac{5}{4}$
(b) Solve for $x: 9x + 3 = 4x - 2$.
 $9x + 3 = 4x - 2 \triangleleft \text{given}$
 $9x = 4x - 5 \triangleleft \text{add } -3 \text{ to both sides}$
 $5x = -5 \triangleleft \text{add } -4x \text{ to both sides}$
 $x = -1 \triangleleft \text{divide both sides by } -1$

Presentation of Answer: x = -1
(c) Solve for
$$x: \frac{1}{2}x = \frac{4}{3}(x-3)$$
.
 $\frac{1}{2}x = \frac{4}{3}(x-3) \triangleleft \text{given}$
 $3x = 8(x-3) \triangleleft \text{multiply both sides by 6}$
 $3x = 8x - 24 \triangleleft \text{expand}$
 $-5x = -24 \triangleleft \text{add } -8x \text{ to both sides}$
 $x = \frac{24}{5} \triangleleft \text{divide both sides by 5}$
Presentation of Answer: $x = \frac{24}{5}$

Exercise 7.2.

7.3. Answers: Make sure you understand the *method* of solution. The answers are given only here.

(a) Solve for x in 5x - 3y = 4.

$$x = \frac{3y+4}{5}$$

(b) Solve for y in 5x - 3y = 4.

$$y = \frac{5x - 4}{3}$$

(c) Solve for z in $x^2z - 12x + y = 1$.

$$\boxed{z = \frac{1 + 12x - y}{x^2}}$$

Exercise 7.3.

7.4. Solutions: (a) Solve $\frac{x}{3x+8} = 5$. $\frac{x}{3x+8} = 5$ x = 5(3x + 8)x = 15x + 40-40 = 14x14x = -40 $x = -\frac{40}{14}$ $x = -\frac{20}{7}$

Presentation of Answer:
$$x = -\frac{20}{7}$$
.

(b) Solve
$$\frac{5x+2}{3-8x} = 2$$
.
 $\frac{5x+2}{3-8x} = 2$ \triangleleft gi
 $5x+2 = 2(3-8x) \triangleleft$ m
 $5x+2 = 6-16x \triangleleft$ ex
 $5x+16x = 6-2 \qquad \triangleleft$ ac
 $21x = 4 \qquad \triangleleft$ co
 $x = \frac{4}{21} \qquad \triangleleft$ di

iven

nultiply both sides by 3 - 8xxpand r.h.s.

dd 16x - 2 to both sides

ombine similar terms

ivide by 21

Presentation of Answer:
$$x = \frac{4}{21}$$
.

(c) Solve
$$(2x - 3)^2 = (2x - 7)^2$$
.
 $(2x - 3)^2 = (2x - 7)^2$ \triangleleft given
 $4x^2 - 12x + 9 = 4x^2 - 28x + 49$ \triangleleft expand using (5), Lesson 5
 $-12x + 9 = -28x + 49$ \triangleleft add $-4x^2$ to both sides
 $28x - 12x = 49 - 9$ \triangleleft add $28x - 9$ to both sides
 $16x = 40$ \triangleleft combine
 $x = \frac{40}{16}$ \triangleleft divide by 16
 $x = \frac{5}{2}$
Presentation of Answer: $x = \frac{5}{2}$.

Comment: This problem is similar to (a) of Example 7.2; however, in my solution I gave a more "traditional' solution. In the second line above, I simply expanded the binomials—this is perhaps what you did yourself. The rest follows using standard methods.

(d) Solve
$$\frac{8x^2 + 2x + 1}{2x + 1} = 4x + 1.$$
$$\frac{8x^2 + 2x + 1}{2x + 1} = 4x + 1$$
$$8x^2 + 2x + 1 = (2x + 1)(4x + 1)$$
$$8x^2 + 2x + 1 = 8x^2 + 6x + 1$$
$$2x + 1 = 6x + 1$$
$$-4x = 0$$
$$x = 0$$

d given
d multiply both sides by 2x + 1
d expand using (2) of Lesson 5
d add -8x² to both sides
d add -6x - 1 to both sides
d divide both sides by -4

Presentation of Answer: x = 0

Now what do you think of that!

Exercise 7.4.

7.5. Solutions: (a) Solve for $x: x^2 - 7x + 12 = 0$. $x^2 - 7x + 12 = 0$ (x-3)(x-4) = 0therefore, either x - 3 = 0 or x - 4 = 0x = 3x = 4Presentation of Solution: x = 3, 4(b) Solve for $x: x^2 + 3x = 10$. $x^2 + 3x = 10$ $x^2 + 3x - 10 = 0$ (x+5)(x-2) = 0therefore, either x + 5 = 0 or x - 2 = 0x = -5x = 2

Presentation of Solution: x = -5, 2

(c) Solve for
$$x: \frac{x^2 + 5x - 6}{x^2 + 1} = 0.$$

$$\frac{x^2 + 5x - 6}{x^2 + 1} = 0$$
$$x^2 + 5x - 6 = 0 \quad \triangleleft \text{ mulitply both sides by } x^2 + 1$$
$$(x + 6)(x - 1) = 0$$

therefore, either

$$x + 6 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -6 \quad x = 1$$
Presentation of Answer: $x = -6, 1$

Exercise 7.5. \blacksquare

7.6. *Answers*:

(a)
$$12x^2 - 17x + 6 = 0$$

$$12x^2 - 17x + 6 = 0 \quad \triangleleft \text{ given}$$
$$(3x - 2)(4x - 3) = 0 \quad \triangleleft \text{ factor it!}$$

From this we can see that the solutions are

$$3x - 2 = 0 \quad \text{or} \quad 4x - 3 = 0$$
$$x = \frac{2}{3} \qquad x = \frac{3}{4}$$
Presentation of Solutions:
$$x = \frac{2}{3}, \frac{3}{4}$$
b)
$$20x^2 + 3x = 2.$$

 $20x^2 + 3x - 2 = 0 \quad \triangleleft \text{ add } -2 \text{ to both sides}$ $(5x + 2)(4x - 1) = 0 \quad \triangleleft \text{ factor it!}$

Thus,

$$5x + 2 = 0 \quad \text{or} \quad 4x - 1 = 0$$

$$x = -\frac{2}{5} \qquad x = \frac{1}{4}$$

$$Presentation \text{ of Solutions:} \quad \boxed{x = -\frac{2}{5}, \frac{1}{4}}$$
(c)
$$\frac{4x^2 - 4x + 1}{x - 1} = 0.$$

$$4x^2 - 4x + 1 = 0 \quad \triangleleft \text{ multiply both sides by } x - 1$$

$$(2x - 1)^2 = 0 \quad \triangleleft \text{ factor it-perfect square!}$$

$$Presentation \text{ of Solution:} \quad \boxed{x = \frac{1}{2}}$$

Exercise 7.6.

7.7. Answers: (a) $x^3 - 2x^2 - 3x = 0$. Factoring this we obtain $x(x^2 - 2x - 3) = 0$ x(x-3)(x+1) = 0Presentation of Solutions: |x = 0, 3, -1|(b) $x^4 - 16 = 0$. Let's factor—difference of two squares! $(x^2 - 4)(x^2 + 4) = 0$ $(x-2)(x+2)(x^2+4) = 0$ Presentation of Solutions: |x = -2, 2|

Comments: The last factor $x^2 + 4$ is an irreducible quadratic—it cannot be factored.

(c) $x^4 - 2x^2 - 3 = 0$. This is a quadratic equation in the variable x^2 : $(x^2)^2 - 2(x^2) - 3 = 0$. If you don't understand what I mean, temporarily put $y = x^2$; our equation becomes $y^2 - 2y - 3 = 0$.

This is clearly a quadratic in y, but $y = x^2$, so it is a quadratic in x^2 . Let's factor it using the factoring techniques.

$$(x^2 - 3)(x^2 + 1) = 0 \triangleleft \text{factor!}$$

 $(x - \sqrt{3})(x + \sqrt{3})(x^2 + 1) = 0 \triangleleft \text{again!}$

My the ZERO-PRODUCT PRINICPLE, we then have

$$x - \sqrt{3} = 0 \quad \text{or} \quad x + \sqrt{3} = 0$$
$$x = \sqrt{3} \qquad x = -\sqrt{3}$$
Presentation of Solutions:
$$x = -\sqrt{3}, \sqrt{3}$$
(d)
$$x^4 - 5x^2 + 6 = 0$$
. This is again quadratic in x^2 .

 $(x^2 - 3)(x^2 - 2) = 0 \quad \triangleleft \text{ factor!}$

$$(x-\sqrt{3})(x+\sqrt{3})(x-\sqrt{2})(x+\sqrt{2}) = 0 \quad \triangleleft \text{ again!}$$

or,

or,

or,

By the ZERO-PRODUCT PRINICPLE, we then have

$$x - \sqrt{3} = 0 \implies x = \sqrt{3}$$

$$x + \sqrt{3} = 0 \implies x = -\sqrt{3}$$

$$x - \sqrt{2} = 0 \implies x = \sqrt{2}$$

$$x + \sqrt{2} = 0 \implies x = -\sqrt{2}$$

Presentation of Solutions:
$$x = -\sqrt{3}, \sqrt{3}, -\sqrt{2}, \sqrt{2}$$

Exercise 7.7. \blacksquare

7.8. Solutions:

(a) Solve for
$$x: 8x^2 - 2x - 1 = 0$$
.

$$8x^{2} - 2x - 1 = 0 \qquad \triangleleft \text{ given}$$

$$8(x^{2} - \frac{1}{4}x) = 1 \qquad \triangleleft \text{ Steps 2 \& 3}$$

$$8(x^{2} - \frac{1}{2}x + \frac{1}{64}) = 1 + \frac{8}{64} \quad \triangleleft \text{ Step 4}$$

$$8(x - \frac{1}{8})^{2} = \frac{9}{8} \qquad \triangleleft \text{ perfect square}$$

$$(x - \frac{1}{8})^{2} = \frac{9}{64} \qquad \triangleleft \text{ divide by 8}$$

$$x - \frac{1}{8} = \pm \frac{3}{8} \qquad \triangleleft \text{ take square root}$$

$$x = \frac{1}{8} \pm \frac{3}{8} \qquad \triangleleft \text{ add } 1/8 \text{ to both sides}$$

Presentation of Solutions: $x = -\frac{1}{4}, \frac{1}{2}$

Comments: Here, my solution uses a slight variation in the techniques illustrated in the examples. Rather than having all term on the left-hand side, I took the constant term to the right-hand

side, then when I completed the square, I added $\frac{8}{64}$ to both sides of the equation.

(b) Solve for $x: 3x^2 + 5x - 2 = 0$.

$$3x^{2} + 5x + 2 = 0 \qquad \triangleleft \text{ given}$$

$$3(x^{2} + \frac{5}{3}x) = -2 \qquad \triangleleft \text{ Steps 2 \& 3}$$

$$3(x^{2} + \frac{5}{3}x + \frac{25}{36}) = -2 + 3(\frac{25}{36}) \qquad \triangleleft \text{ Step 4}$$

$$3(x + \frac{5}{6})^{2} = \frac{1}{12} \qquad \triangleleft \text{ perfect square}$$

$$(x + \frac{5}{6})^{2} = \frac{1}{36} \qquad \triangleleft \text{ divide by 8}$$

$$x + \frac{5}{6} = \pm \frac{1}{6} \qquad \triangleleft \text{ take square root}$$

$$x = -\frac{5}{6} \pm \frac{1}{6} \qquad \triangleleft \text{ add } 1/8 \text{ to both sides}$$

Presentation of Solution: $x = -\frac{2}{3}, -1$

(c) Solve for $x: x^2 + x - 1 = 0$. $x^2 + x - 1 = 0$ ⊲ given $x^2 + x = 1$ **⊲** Steps 2 & 3 $(x^2 + x + \frac{1}{4}) = 1 + \frac{1}{4}$ ⊲ Step 4 $(x+\frac{1}{2})^2 = \frac{5}{4}$ ⊲ perfect square $x + \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$ ⊲ take square root $x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \triangleleft \text{add} -\frac{1}{2}$ to both sides Presentation of Solution: $\left| x = -\frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{1}{2} - \frac{\sqrt{5}}{2} \right|$ or, Presentation of Solution: $x = \frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}+1}{2}$ (Verify!) Exercise 7.8.

7.9. Solutions: (a) Solve for $x: 2x^2 + 5x - 12 = 0$. $x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-12)}}{2(2)} \quad \triangleleft a = 2, b = 5, c = -12$ $=\frac{-5\pm\sqrt{121}}{4}$ $= \frac{-5 \pm 11}{4} \\ = \frac{3}{2}, -4$

Presentation of Solution: $x = \frac{3}{2}, -4$

(b) Solve for
$$x: 3x^2 - 7x + 1 = 0$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(1)}}{2(3)} \quad \triangleleft a = 3, b = -7, c = 1$$

$$= \frac{7 + \sqrt{37}}{6}, \frac{7 - \sqrt{37}}{6}$$
Presentation of Solutions: $x = \frac{7 + \sqrt{37}}{6}, \frac{7 - \sqrt{37}}{6}$
(c) Solve for $x: x^2 + 1 = 0$

$$b^{2} - 4ac = 0^{2} - 4(1)(1) = -4 < 0 \quad \triangleleft a = 1, b = 0, c = 1$$

Therefore, this equation has no solutions.

(d) Solve for $x: x^2 + x = 3$. Begin by putting it into the proper form: $x^2 + x - 3 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)(-3)}}{2} \quad \triangleleft a = 1, b = 1, c = -3$$
$$= \frac{-1 \pm \sqrt{13}}{2}$$

Presentation of Solutions:
$$x = \frac{-1 + \sqrt{13}}{2}, \frac{-1 - \sqrt{13}}{2}$$

Exercise 7.9. \blacksquare

7.10. Solution: Solve for y in $yx^2 - 2x - 4y^2 = 0$. The first thing to do is to rearrange the equation:

$$-4y^2 + x^2y - 2x = 0$$

Now we can see that a = -4, $b = x^2$, and c = -2x.

$$y = \frac{-x^2 \pm \sqrt{(x^2)^2 - 4(-4)(-2x)}}{2(-4)}$$
$$= \frac{-x^2 \pm \sqrt{x^4 - 32x}}{(-8)} = -\frac{-x^2 \pm \sqrt{x^4 - 32x}}{8}$$
$$= \frac{x^2 \pm \sqrt{x^4 - 32x}}{8}$$

Presentation of Solutions:

$$y = \frac{x^2 + \sqrt{x^4 - 32x}}{8}, \quad \frac{x^2 - \sqrt{x^4 - 32x}}{8}$$

Comments: We get real solutions when x is such that $x^4 - 32x \ge 0$. This problem will be continued in the section on Solving Inequalities.

Exercise 7.10.

7.11. Solution to (a): Begin by rearranging the equation in the form of a quadratic in w:

$$5wx^4 - 2w^2x^2 + 3 = 0$$

becomes

$$(-2x^2)w^2 + (5x^4)w + 3$$

Now it is apparent that $a = -2x^2$, $b = 5x^4$, and c = 3.

$$w = \frac{-5x^4 \pm \sqrt{(5x^4)^2 - 4(-2x^2)(3)}}{2(-2x^2)}$$
$$= -\frac{-5x^4 \pm \sqrt{25x^8 + 24x^2}}{4x^2}$$
$$= -\frac{-5x^4 \pm |x|\sqrt{25x^6 + 24}}{4x^2}$$

Now here's an interesting simplification. How to remove the absolute value? If $x \ge 0$, the |x| = x; if x < 0, then |x| = -x. The second case has the effect of changing '±' (plus or minus) to ' \mp ' (minus or plus).

So I think it is safe to remove the absolute value—it will not change the solutions. Therefore,

$$w = -\frac{-5x^4 \pm x\sqrt{25x^6 + 24}}{4x^2}$$
$$= -\frac{-5x^3 \pm \sqrt{25x^6 + 24}}{4x}$$

Now x is a factor common to both the numerator and denominator—we can and do cancel it.

Presentation of Answer:
$$w = -\frac{-5x^3 \pm \sqrt{25x^6 + 24}}{4x}$$

Did you get it?

Solution to (b): Solve for x^2 in $5wx^4 - 2w^2x^2 + 3 = 0$.

$$x^{2} = \frac{-(-2w^{2}) \pm \sqrt{(-2w^{2})^{2}} - 4(5w)(3)}{2(5w)}$$
$$= \frac{2w^{2} \pm \sqrt{4w^{4} - 60w}}{10w}$$
$$= \frac{2w^{2} \pm 2\sqrt{w^{4} - 15w}}{10w}$$
$$= \frac{w^{2} \pm \sqrt{w^{4} - 15w}}{5w}$$

Presentation of Solution:

$$x^{2} = \frac{w^{2} \pm \sqrt{w^{4} - 15w}}{5w}$$
(A-1)

Comments: Real solutions exist provided $w^4 - 15w \ge 0$. When we take up solving inequalities, we'll return to this problem to do a more

detailed analysis of it. Suffice it to say that solutions exist for x^2 provided (1) $w^4 - 15w \ge 0$ and (2) the right-hand side of (A-1) must be nonnegative as well.

Exact conditions under which this equation has solution for x^2 can be made after we analyze some inequalities ... later. Exercise 7.11.

7.12. Solutions:

(a) Solve for $x: 4x + 12 \le 5$.

$$\begin{array}{ll} 4x+12 \leq 5 & \triangleleft \mbox{ given} \\ & 4x \leq -7 & \triangleleft \mbox{ add } -12 \mbox{ to both sides } (10) \\ & x \leq -\frac{7}{4} & \triangleleft \mbox{ multiply both sides by } 1/4\mbox{: see } (11) \end{array}$$

Presentation of Answer. Interval Notation:

$$\left(-\infty,-\frac{7}{4},
ight]$$

(b) Solve for $x: \frac{1}{2}x - \frac{3}{2} > 4$. $\frac{1}{2}x - \frac{3}{2} > 4 \quad \triangleleft \text{ given}$ $x - 3 > 8 \quad \triangleleft \text{ multiply both sides by 2: (11)}$ $x > 11 \quad \triangleleft \text{ add 3 to both sides: see: (10)}$

Presentation of Answer. Interval Notation: $|(11, +\infty)|$

(c) Solve for
$$x: 3 - 8x \ge 4$$

 $3 - 8x \ge 4 \quad \triangleleft \text{ given}$
 $-8x \ge -1 \quad \triangleleft \text{ add } -3 \text{ to both sides: (10)}$
 $x \le \frac{1}{8} \quad \triangleleft \text{ inequality reversed! See: (12)}$

Presentation of Answer. Interval Notation:

$$\left(-\infty, \frac{1}{8}\right]$$

Exercise 7.12. \blacksquare

7.13. Solutons: (a) Solve for $x: 2x - 1 \le 5x + 2$. $2x - 1 \le 5x + 2 \triangleleft \text{given}$ -3x < 3 \triangleleft add 1 - 5x to both sides $x \ge -1$ \triangleleft inequality reversed: (12) Presentation of Answer: $|\{x \mid x \geq -1\}|$ (b) Solve for x: 3x + 4 > 1 - 6x. $3x + 4 > 1 - 6x \triangleleft \text{given}$ 9x > -3 \triangleleft add 6x - 4 to both sides: (10) $x > -\frac{1}{2}$ \triangleleft inequalities not reversed: (10) Presentation of Answer: $\left\{ x \mid x > -\frac{1}{3} \right\}$

Exercise 7.13.

7.14. Solutions: (a) Solve for x: 3 < 2x + 1 < 12. $3 < 2x + 1 \leq 12$ \triangleleft given 2 < 2x < 11 \triangleleft add -1 to all sides $1 < x \leq \frac{11}{2}$ \triangleleft multiply 1/2 on all sides Presentation of Answer: $\left| \left(1, \frac{11}{2} \right] \right|$ (b) Solve for $x: -2 \le 2 - 5x \le -1$. $-2 \leq 2 - 5x \leq -1$ d given -4 < -5x < -3 \triangleleft add -2 to all sides $\frac{4}{5} \ge x \ge \frac{3}{5}$ Presentation of Answer: $\left[\left[\frac{4}{5}, \frac{3}{5} \right] \right]$

(c) Solve for
$$x: 1 < \frac{3}{2}x + 1 < 4$$
.
 $1 < \frac{3}{2}x + 1 < 4 \quad \triangleleft \text{ given}$
 $2 < 3x + 2 < 8 \quad \triangleleft \text{ multiply all sides by 2}$
 $0 < 3x < 6 \quad \triangleleft \text{ add } -2 \text{ to all sides}$
 $0 < x < 2 \quad \triangleleft \text{ multiply all sides by 1/3}$

Presentation of Answer: (0,2)

Exercise 7.14.

7.15. Solutions to(a): Solve for $x: x^2 - x - 2 \le 0$. Begin by factoring the quadratic.

$$(x+1)(x-2) \le 0$$

Now, do a Sign Chart Analysis.



Comments: We include the endpoints in the solution because they would satisfy the inequality—that's a good reason!

Solution to (b): Solve for $x: x^3 - 4x^2 + 3x > 0$. Begin by factoring completely:

$$x(x-1)(x-3) > 0.$$

x0 x-11 ----- x - 33 ---- x(x-1)(x-3)0 1 3 *leqend*: • negative (-)• positive (+)Presentation of Solution. Interval Notation: $|(0,1) \cup (3,+\infty)|$ **Set Notation:** $| \{ x | 0 < x < 1 \text{ or } x > 3 \}$

The Sign Chart of x(x-1)(x-3)

Comments: Here, we *do not* include the endpoints. At the endpoints, the expression is zero, which does not satisfy the inequality.

Exercise 7.15. \blacksquare

7.16. Solution to (a): Solve for x in $x^2 - 3x \le 4$. Begin by taking the 4 to the left-hand side of the inequality,

$$x^2 - 3x - 4 \le 0.$$

Now factor,

$$(x+1)(x-4) \le 0.$$

Now do a Sign Chart for (x+1)(x-4).



Solution to (b): Solve for x in $6x - x^2 < 5$. Therefore,

$$\begin{array}{ll} 6x-x^2 < 5 & \triangleleft \mbox{ given} \\ 6x-x^2-5 < 0 & \triangleleft \mbox{ put everything to l.h.s.} \\ x^2-6x+5 > 0 & \triangleleft \mbox{ multiply by -1: I didn't like the } -x^2 \\ (x-1)(x-5) > 0 & \triangleleft \mbox{ now factor!} \end{array}$$

Now do a Sign Chart for (x-1)(x-5).





Presentation of Solution: Solve
$$\frac{x(x-2)}{x+1} < 0.$$
Set Notation: $\{x \mid x < -1 \text{ or } 0 < x < 2\}$ Interval Notation: $(-\infty, -1) \cup (0, 2)$

Solution to (b) is on the next page.
Solution to (b): Solve for
$$x$$
: $\frac{x}{x^2 - 3x + 2} \ge 0$.

Begin by factoring,

$$\frac{x}{(x-1)(x-2)} \ge 0$$

It is this inequality we solve using the Sign Chart Method.



Presentation of Solution: Solve
$$\frac{x}{(x-1)(x-2)} \ge 0.$$
Set Notation: $\{x \mid 0 \le x < 1 \text{ or } x > 2\}$ Interval Notation: $[0,1) \cup (2,+\infty)$

We include x = 0 in the solution because this point makes the expression 0; hence, x = 0 satisfies the inequality. We *don't* include x = 1 or x = 2 in the solution set because they make the denominator equal to zero—a no-no. Exercise 7.17.

7.18. Solution: Solve $x^4 - 32x \ge 0$. First factor: $x(x^3 - 32) \ge 0$.

The second factor can be factored some more (it is the difference in cubes), but in this instance it would be a waste of pencil lead to do so. The important point about the factors is that you can discern when they are positive and when they are negative—we can do that for the factor $(x^3 - 32)$.

We must ask the question, when is $x^3 - 32 > 0$? The Answer: when $x^3 > 32$ or when $x > \sqrt[3]{32}$. Similarly, $x^3 - 32 < 0$ when $x < \sqrt[3]{32}$. Thus, ...



Presentation of Solution: Solve for $x: x^4 - 32x \ge 0$.

Interval Notation: $(-\infty, 0] \cup [\sqrt[3]{32}, +\infty)$

Comments: This finally answers the question: For what values of x does the equation

$$yx^2 - 2x - 4y^2 = 0$$

have real solutions for y? The answer is that x must be a number less than or equal to 0 or a number greater than or equal to $\sqrt[3]{32}$. Otherwise, there are no real solutions for y. Exercise 7.18.

7.19. Solution: Using the Method of Signs we can easily solve the inequality $w^4 - 15w \ge 0$ in exactly the same way we did in EXER-CISE 7.18. Thus,

$$w^4 - 15w \ge 0$$
 has solution $w \le 0$ or $w \ge \sqrt[3]{15}$

To continue the analysis, we need to break our argument down into cases.

Case 1. $w \ge \sqrt[3]{15}$, in particular w is *positive*.

Since w > 0, $w^4 - 15w < w^4$. Take the square root of both sides we get $\sqrt{w^4 - 15w} < w^2$. This implies that

$$w^2 \pm \sqrt{w^4 - 15w} > 0$$

since we are adding/subtracting a smaller number. But this means,

$$x^2 = \frac{w^2 \pm \sqrt{w^4 - 15w}}{5w} > 0$$

since both numerator and denominator are positive.

Case 2. $w \leq 0$. Actually, $w \neq 0$ since w appears in the denominator.

Since w < 0, $w^4 < w^4 - 15w$, since we are subtracting a *negative number*, the result will be *larger*. Taking roots, we get $w^2 < \sqrt{w^4 - 15w}$. Therefore,

$$w^{2} + \sqrt{w^{4} - 15w} > 0$$
 but $w^{2} - \sqrt{w^{4} - 15w} < 0$

This means,

$$\frac{w^2 + \sqrt{w^4 - 15w}}{5w} < 0 \text{ and } \frac{w^2 - \sqrt{w^4 - 15w}}{5w} > 0$$

since, you'll recall, we are assuming in this case that w < 0.

Summary.

1. For any $w \ge \sqrt[3]{15}$,

$$\frac{w^2\pm\sqrt{w^4-15w}}{5w}>0$$

In this case, there are *four solutions* for x.

$$x^{2} = \frac{w^{2} \pm \sqrt{w^{4} - 15w}}{5w} \implies x = \pm \sqrt{\frac{w^{2} \pm \sqrt{w^{4} - 15w}}{5w}}$$

Ouch!

2. For any w < 0, only

$$\frac{w^2 - \sqrt{w^4 - 15w}}{5w} > 0.$$

In this case, there are two solutions for x.

$$x^{2} = \frac{w^{2} - \sqrt{w^{4} - 15w}}{5w} \implies x = \pm \sqrt{\frac{w^{2} - \sqrt{w^{4} - 15w}}{5w}}$$

That was ugly!

Exercise 7.19.

7.20. Solutions: (a) Solve for x: |x+3| < 8. $|x+3| < 8 \triangleleft \text{given}$ -8 < x + 3 < 8 \triangleleft from (16) -11 < x < 5 \triangleleft add -3 to all sides Presentation of Solution: (-11, 5)(b) Solve for x: |4x + 9| < 1. $|4x+9| \le 1 \quad \triangleleft \text{ given}$ $-1 \leq 4x + 9 \leq 1 \quad \triangleleft \text{ from (16)}$ $-10 < 4x < -8 \triangleleft \text{add} -9 \text{ to all sides}$ $-\frac{10}{4} < x \leq -\frac{8}{4}$

 multiply all sides by 1/4

Now, reducing to lowest terms we get the ...

Presentation of Solution: $\left[-\frac{5}{2}, -2\right]$

(c) Solve for $x: |2 - 7x| \le 3$. $|2 - 7x| \le 3 \quad \triangleleft \text{ given}$ $-3 \le 2 - 7x \le 3 \quad \triangleleft \text{ from (16)}$ $-5 \le -7x \le 1 \quad \triangleleft \text{ add } -2 \text{ to all sides}$ $\frac{5}{7} \ge x \ge -\frac{1}{7} \quad \triangleleft \text{ multiply all sides by } -1/7$ or

$$-\frac{1}{7} \le x \le \frac{5}{7}$$

In the last step we have multiplied both sides by a negative number, this will *reverse the direction of the inequality*!

Presentation of Solution:
$$\left[-\frac{1}{7}, \frac{5}{7}\right]$$

Exercise 7.20.

7.21. Solution to (a) Solve for $x: |9x - 2| \ge 3$.

$$|9x - 2| \ge 3$$

Use (17) to **split** the inequality!

Now, join the solutions!

Solution Set
$$= \left[\frac{5}{9}, +\infty\right) \cup \left(-\infty, -\frac{1}{9}\right]$$

Presentation of Solution: $\left[\left(-\infty, -\frac{1}{9}\right] \cup \left[\frac{5}{9}, +\infty\right)\right]$

Solution to (b) Solve for x: |2 - 3x| > 6.

$$|2 - 3x| > 6$$

Use (17) to **split** the inequality!

2 - 3x > 6	\triangleleft upper inequality	2 - 3x < -6	\triangleleft lower inequality
-3x > 4	\triangleleft add -2	-3x < -8	\triangleleft add -2
$x < -\frac{4}{3}$	⊲ divide by -3	$x > \frac{8}{3}$	\triangleleft divide by 5
$(-\infty,-rac{4}{3})$	\triangleleft solution set	$(rac{8}{3},+\infty)$	\triangleleft solution set

Now, **join** the solutions!

Solution Set =
$$(-\infty, -\frac{4}{3}) \cup (\frac{8}{3}, +\infty)$$

Presentation of Solution: $(-\infty, -\frac{4}{3}) \cup (\frac{8}{3}, +\infty)$

Comment: Hopefully, you understand when to include the endpoints and when not to include them in your solution set, and, most importantly, how do *denote* the inclusion/exclusion of the endpoints.

Solution to (c) Solve for
$$x: |\frac{3}{2}x + 2| > \frac{1}{3}$$
.
 $|\frac{3}{2}x + 2| > \frac{1}{3}$

Use (17) to **split** the inequality!

$\frac{3}{2}x + 2 > \frac{1}{3}$	\triangleleft upper inequality	$\frac{3}{2}x + 2 < -\frac{1}{3}$	\triangleleft lower inequality
9x + 12 > 2	\triangleleft multiply by 6	9x + 12 < -2	\triangleleft multiply by 6
9x > -10	\triangleleft add -12	9x < -14	\triangleleft add -12
$x > -\frac{10}{9}$	\triangleleft divide by 9	$x < -\frac{14}{9}$	\triangleleft divide by 9
$\left(-\frac{10}{9},+\infty\right)$	\triangleleft solution set	$(-\infty,-\frac{14}{9})$	\triangleleft solution set

Now, join the solutions!

Solution Set =
$$\left(-\frac{10}{9}, +\infty\right) \cup \left(-\infty, -\frac{14}{9}\right)$$

Presentation of Solution: $\left[\left(-\infty, -\frac{14}{9}\right) \cup \left(-\frac{10}{9}, +\infty\right)\right]$ Exercise 7.21. •

Solutions to Examples

7.1.

(a) Solve for x: 4x + 5 = 0Solution:

4x + 5 = 0	⊲ given
(4x+5) - 5 = -5	\triangleleft subtract -5 from both sides
4x = -5	⊲ combine
$\frac{1}{4}(4x) = \frac{1}{4}(-5)$	\triangleleft multi. both sides by 1/4
$x = -\frac{5}{4}$	\triangleleft combine and done!

Solving an equation consists of a sequence of *legal* steps: adding the same quantity to both sides (see equation (1)); multiplying both sides by the same quantity (see equation (2)) were used above.

Presentation of Answer:
$$x = -\frac{5}{4}$$
.
(b) Solve for $x: \frac{1}{2}x - 4 = 6$.
Solution:
 $\frac{1}{2}x - 4 = 6$ \triangleleft given
 $(\frac{1}{2}x - 4) + 4 = 6 + 4 \quad \triangleleft \text{ add } 4 \text{ to both sides}$
 $\frac{1}{2}x = 10 \quad \triangleleft \text{ combine}$
 $2(\frac{1}{2}x) = 2(10) \quad \triangleleft \text{ multi. both sides by 2}$
 $x = 20 \quad \triangleleft \text{ combine and done!}$

Note this equation initially was not of the form ax + b = 0. There was a nonzero constant on right-hand side of the equation. Obviously, the equation $\frac{1}{2}x - 4 = 6$ is equivalent to $\frac{1}{2}x - 10 = 0$; consequently, no big deal was made of it.

Presentation of Solution: x = 20.

(c) Solve for
$$x: 7 - 3x = 2$$
.
Solution:

$$7 - 3x = 2 \qquad \triangleleft \text{ given}$$

$$-(7 - 3x) = -2 \qquad \triangleleft \text{ multiply both sides by } -1$$

$$3x - 7 = -2 \qquad \triangleleft \text{ simplify}$$

$$(3x - 7) + 7 = -2 + 7 \qquad \triangleleft \text{ add } 7 \text{ to both sides}$$

$$3x = 5 \qquad \triangleleft \text{ combine}$$

$$\frac{1}{3}(3x) = \frac{1}{3}(5) \qquad \triangleleft \text{ multiply both sides by } 1/3$$

$$x = \frac{5}{3} \qquad \triangleleft \text{ combine and done!}$$

Comments: Here the coefficient of x was negative. I opted to multiply by -1 to 'change the sign' so that I would feel more comfortable solving.

Presentation of Answer:
$$x = \frac{5}{3}$$
.

Example 7.1. \blacksquare

7.2. Solutions:

(a) Solve for $x: (x - 1)^2 = x^2$.

$$(x-1)^2 = x^2 \quad \triangleleft \text{ given}$$

$$(x-1)^2 - x^2 = 0 \quad \triangleleft \text{ sub. } x^2 \text{ both sides: see (1)}$$

$$[(x-1)-x][(x-1)+x] = 0 \quad \triangleleft \text{ diff. of squares!}$$

$$-(2x-1) = 0 \quad \triangleleft \text{ combine}$$

$$2x-1 = 0 \quad \triangleleft \text{ multipy by -1: see (2)}$$

$$2x = 1 \quad \triangleleft \text{ add 1 to both sides: see (1)}$$

$$x = \frac{1}{2} \quad \triangleleft \text{ divide by 2: see (3)}$$

Comments: Note the *recommended* way of handling an equation with squares on both sides. Rather than taking the square root of both sides, take everything to one side of the equation and factor it as a *difference of squares*: this is a much nicer method. The other option is to *expand* all binomials and combine.

$$Presentation of Answer: \boxed{x = \frac{1}{2}}$$
(b) Solve for $x: \frac{2x^2 + 5x - 1}{x + 1} = 2x + 1$

$$\frac{2x^2 + 5x - 1}{x + 1} = 2x + 1 \qquad \triangleleft \text{ given equation}$$

$$2x^2 + 5x - 1 = (x + 1)(2x + 1) \qquad \triangleleft \text{ multiply both sides by } x + 1$$

$$2x^2 + 5x - 1 = 2x^2 + 3x + 1 \qquad \triangleleft \text{ expand r.h.s.}$$

$$5x - 1 = 3x + 1 \qquad \triangleleft \text{ sub. } 2x^2 \text{ from both sides: (1)}$$

$$2x = 2 \qquad \triangleleft \text{ two steps in one!}$$

$$x = 1 \qquad \triangleleft \text{ divide both sides by } 2$$

Presentation of Answer:
$$x = 1$$

(c) Solve for
$$x: \frac{1-3x}{2x+1} = 4$$
.

$$\frac{1-3x}{2x+1} = 4 \qquad \triangleleft \text{ given equation}$$

$$1-3x = 4(2x+1) \qquad \triangleleft \text{ multiply both sides by } 2x+1$$

$$1-3x = 8x+4 \qquad \triangleleft \text{ expand r.h.s.}$$

$$-11x = 3 \qquad \triangleleft \text{ add } -8x-1 \text{ to both sides}$$

$$x = -\frac{3}{11} \qquad \triangleleft \text{ divide both sides by } 11$$

Presentation of Answer:
$$x = -\frac{3}{11}$$

(d) Solve for
$$x: \frac{4}{3x+1} = 1$$
.

$$\frac{4}{3x+1} = 1 \qquad \triangleleft \text{ the given equation}$$

$$4 = 3x+1 \quad \triangleleft \text{ multiply both sides by } 3x+1$$

$$3x+1 = 4 \qquad \triangleleft \text{ transpose equation: } a = b \iff b = a$$

$$3x = 3 \qquad \triangleleft \text{ add } -1 \text{ to both sides}$$

$$x = 1 \qquad \triangleleft \text{ divide both sides by } 3$$

Presentation of Answer: x = 1

Example 7.2. \blacksquare

- **7.3.** Solutions:
 - (a) Solve $x^2 5x + 6 = 0$. Solution:

$$x^{2} - 5x + 6 = 0 \quad \triangleleft \text{ given}$$
$$(x - 2)(x - 3) = 0 \quad \triangleleft \text{ factor l.h.s.}$$
(S-1)

From (S-1) and the ZERO-PRODUCT PRINCIPLE, we deduce that either

$$x - 2 = 0$$
 or $x - 3 = 0$
 $x = 2$ $x = 3$

Presentation of Solution: x = 2, 3.

Comments: Note that this equation has **two distinct solutions.** An alternate method of presenting the solutions to an equation is to present the **solution set**: the set of all solutions. Here we could have written,

Solution Set = $\{2,3\}$

(b) Solve
$$x^2 + 4x + 4 = 0$$
.
Solution:
 $x^2 + 4x + 4 = 0$ \triangleleft given equation
 $(x+2)^2 = 0$ \triangleleft factor: perfect square!
Therefore,
 $(x+2) = 0$ \triangleleft Zero-Product Principle
or,
 $x = -2$

Presentation of Answer: x = -2

Comments: The left-hand side is a quadratic polynomial that, it turned out, was a perfect square. Note that this equation has only **one solution**.

(c) Solve
$$6x^2 - x - 2 = 0$$
.

Solution: Let's take the polynomial aside and factor it using the factoring techniques explained in LESSON 6.

Factor: $6x^2 - x - 2 = 0$. Begin by trying a factorization of the form

$$6x^2 - x - 2 = (3x + r_1)(2x + r_2),$$

where r_1 and r_2 are chosen so that $r_1r_2 = -2$ and the "crossproduct" is correct. After some trial and error we get $r_1 = -2$ and $r_2 = 1$; thus,

$$6x^{2} - x - 2 = (3x - 2)(2x + 1).$$
 (S-2)

Solve: Now let's take the factorization (S-2) and return to our original equation:

$$(3x-2)(2x+1) = 0$$

We can now obtain the solutions by applying the Zero-Product Principle: (3x - 2)(2x + 1) = 0 implies either ...

$$3x - 2 = 0 \quad \text{or} \quad 2x + 1 = 0$$
$$3x = 2 \quad 2x = -1$$
$$x = \frac{2}{3} \quad x = -\frac{1}{2}$$

We deduce the solution set is $\left\{\frac{2}{3}, -\frac{1}{2}\right\}$.

Presentation of Solution:
$$x = \frac{2}{3}, -\frac{1}{2}$$

Example 7.3. \blacksquare

7.4. Solution: This will be short. All the work has been already done.

$$2x^{2} + 12x - 3 = 0 \qquad \triangleleft \text{ given}$$

$$2(x+3)^{2} - 21 = 0 \qquad \triangleleft \text{ complete the square}$$

$$2(x+3)^{2} = 21 \qquad \triangleleft \text{ equation of the form (7)}$$

$$(x+3)^{2} = \frac{21}{2} \qquad \triangleleft \text{ divide by 2}$$

$$x+3 = \pm \sqrt{\frac{21}{2}} \qquad \triangleleft \text{ take square root}$$

$$x = -3 \pm \sqrt{\frac{21}{2}} \qquad \triangleleft \text{ solved!}$$
Presentation of Solution:
$$x = -3 \pm \sqrt{\frac{21}{2}}, -3 - \sqrt{\frac{21}{2}}$$

Example 7.4. \blacksquare

7.5. Solution: Just follow the algorithm.

$$3x^{2} + 2x - 5 = 0 \qquad \triangleleft \text{ given}$$

$$3(x^{2} + \frac{2}{3}x) - 5 = 0 \qquad \triangleleft \text{ Step 2 \& 3}$$

$$3(x^{2} + \frac{2}{3}x + \frac{1}{9}) - 5 - \frac{1}{3} = 0 \qquad \triangleleft \text{ Step 4: add/sub 1/2 coeff. of } x$$

$$3(x + \frac{1}{3})^{2} = \frac{16}{3} \qquad \triangleleft \text{ perfect square!}$$

$$(x + \frac{1}{3})^{2} = \frac{16}{9} \qquad \triangleleft \text{ divide by 3}$$

$$x + \frac{1}{3} = \pm \frac{4}{3} \qquad \triangleleft \text{ take square root both sides}$$

$$x = -\frac{1}{3} \pm \frac{4}{3} \qquad \triangleleft \text{ solve for } x \text{ and done}$$

Thus,

$$x = -\frac{1}{3} \pm \frac{4}{3}$$

= $-\frac{1}{3} + \frac{4}{3}, -\frac{1}{3} - \frac{4}{3}$
= $1, -\frac{5}{3}$
 $x = -\frac{5}{3}, 1$

Example 7.5. \blacksquare

Presentation of Solution:

- 7.6. Solutions: We simply apply the QUADRATIC FORMULA!
- (a) Solve for $x: x^2 5x + 6 = 0$. This is the case where a = 1, b = -5, and c = 6:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{25 - 24}}{2}$$
$$= \frac{5 \pm 1}{2}$$
$$= \frac{5 \pm 1}{2}, \frac{5 - 1}{2}$$
$$= \frac{6}{2}, \frac{4}{2}$$
$$= 3, 2$$

Presentation of Answers: x = 2, 3

(b) Solve for $x: x^2 + 4x + 4 = 0$. This is the case where a = 1, b = 4, and c = 4.

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)}$$
$$= \frac{-4 \pm \sqrt{16 - 16}}{2}$$
$$= \frac{-4}{2}$$
$$= -2$$

Presentation of Solution: x = -2

Comments: This is the case of the discriminant of zero. Therefore we have only one solution.

(c) Solve for
$$x: 6x^2 - x - 2 = 0$$
. Here $a = 6, b = -1$, and $c = -2$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-2)}}{2(6)}$$

$$= \frac{1 \pm \sqrt{1 + 48}}{12}$$

$$= \frac{1 \pm 7}{12}$$

$$= \frac{1 \pm 7}{12}, \frac{1 - 7}{12}$$

$$= \frac{8}{12}, \frac{-6}{12}$$

$$= \frac{2}{3}, -\frac{1}{2}$$
Presentation of Solutions: $x = \frac{2}{3}, -\frac{1}{2}$

(d) Solve for $x: 3x^2 - 3x + 1 = 0$. This is the case of a = 3, b = -3 and c = 1. Before starting the standard calculations, we might check the discriminant:

$$b^{2} - 4ac = (-3)^{2} - 4(3)(1) = 9 - 12 = -3 < 0.$$

We've saved ourselves some work. This equation has no solutions.

Example 7.6.

7.7. Solve $1 + 4y^3 \ge 0$. We simply use the tools:

$$\begin{array}{ll} 1+4y^3 \geq 0 & \triangleleft \mbox{ given} \\ 4y^3 \geq -1 & \triangleleft \mbox{ add } -1 \mbox{ both sides: see (10)} \\ y^3 \geq -\frac{1}{4} & \triangleleft \mbox{ multiply 1/4 both sides: see (11)} \\ \sqrt[3]{y^3} \geq \sqrt[3]{-\frac{1}{4}} & \triangleleft \mbox{ by (13)} \\ y \geq -\frac{1}{\sqrt[3]{4}} & \triangleleft \mbox{ apply properties of radicals: see (1)} \end{array}$$

The answer can be presented in three ways.

- 1. Inequalities. Solution: $y \ge -\frac{1}{\sqrt[3]{4}}$.
- 2. Interval Notation. Solution: $\left[-\frac{1}{\sqrt[3]{4}},\infty\right)$.

3. Set Notation. Solution: $\left\{ y \mid y \ge -\frac{1}{\sqrt[3]{4}} \right\}$.

Example 7.7.

7.8. Solutions: (a) Solve for x: 5x + 7 < 0.

 $\begin{array}{ll} x+7 < 0 & \triangleleft \mbox{ given} \\ 5x < -7 & \triangleleft \mbox{ add } -7 \mbox{ to both sides } (10) \\ x < -\frac{7}{5} & \triangleleft \mbox{ multiply both sides by 1/5: see } (11) \end{array}$

Presentation of Answer. Interval Notation:

$$\left(-\infty,-\frac{7}{5}\right)$$

(b) Solve for $x \ 3 - 9x \ge 4$.

$$\begin{array}{lll} 3-9x \geq 4 & \triangleleft \mbox{ given} \\ -9x \geq 1 & \triangleleft \mbox{ add } -3 \mbox{ to both sides} \\ & x \leq -\frac{1}{9} & \triangleleft \mbox{ mul. by } -1/9, \mbox{ inequality reversed! See (12)} \end{array}$$

Presentation of Answer. Set Notation: $\left\{ x \mid x \leq -\frac{1}{9} \right\}$

(c) Solve for
$$x: 3x^5 + 4 \ge 9$$
.
 $3x^5 + 4 \ge 9 \quad \triangleleft \text{ given}$
 $3x^5 \ge 5 \quad \triangleleft \text{ add } -4 \text{ to both sides}$
 $x^5 \ge \frac{5}{3} \quad \triangleleft \text{ multiply by } 1/3$
 $x \ge \sqrt[5]{\frac{5}{3}} \quad \triangleleft \text{ take 5th root both sides: See (1)}$

Presentation of Answer. Inequalities. $x \ge \sqrt[5]{\frac{5}{3}}$ (d) Solve for $x: 3x^2 + 4 \le 3$.

 $3x^2 + 4 \le 3 \qquad \triangleleft \text{ given}$ $3x^2 \le -1 \quad \triangleleft \text{ add } -4 \text{ to both sides}$ $x^2 \le -\frac{1}{3} \qquad \triangleleft \text{ divide by } 3$

Now it is apparent that there are no solutions to this equation. For any $x, x^2 \ge 0$ so x^2 cannot be less than a negative number.

Presentation of Solution. Set Notation. The solution set is \emptyset , the *empty set*.

Example 7.8.

7.9. Solution to (a): This method begins by factoring the algebraic expressions:

$$(x-1)(x-2) \ge 0$$
 (S-3)

The idea is to analyze when each factor is positive and when each factor is negative. The **Sign Chart** is a graphical scheme for storing all the information. Here's the **Sign Chart** for (x - 1)(x - 2):

The Sign Chart of (x-1)(x-2)



Sign Chart of x - 1. We ask the question, where is x - 1 > 0? Answer: x > 1. We mark that in blue on the axis. (When using pencil and paper, I write little '+' signs over the axis.) We now ask the

question, where is x - 1 < 0? Answer: x < 1. We mark that in red on the axis. (Pencil and paper? Write little '-' signs over the axis.) And, of course, x - 1 = 0 when x = 1. This represents a complete analysis of the factor x - 1: when its positive, when its negative, and when its zero.

Sign Chart of x - 2. We do the same analysis for the factor x - 2 indicating the results in blue (or a '+') or in red (or a '-').

Sign Chart of (x - 1)(x - 2). To obtain the final sign chart for the expression of interest, we put the other charts together using the following principles:

$$(-)(-) = (+), (+)(-) = (-) \text{ and } (+)(+) = (+)$$

For example, if we consider a number x < 1, then, according to my sign charts, the first factor (x - 1) is negative and the second factor (x - 2) is negative too; the product (x - 1)(x - 2) is positive and we indicate that on the sign chart for (x - 1)(x - 2). Get the idea?

The last axis tells us exactly when (x-1)(x-2) is positive and when it is negative. We can see that the solution to the inequality

$$x^2 - 3x + 2 \ge 0$$

is all x in the blue. Members of the solution set are all $x \leq 1$ plus all $x \geq 2$. (We include x = 1 and x = 2 because they make the expression $x^2 - 3x + 2 = 0$ and would therefore satisfy the inequality.)

Presentation of Solution:

Set Notation:
$$\{x \mid x \le 1 \text{ or } x \ge 2\}$$

Interval Notation: $(-\infty, 1] \cup [2, +\infty)$

Solution to: (b) Not as much detail will be given in this solution.

Problem: Solve for x in $x < x^2$. The first thing we must do is to take all quantities to one side of the inequality: $x - x^2 < 0$. Next, we must factor, x(1-x) < 0. Finally, we apply our Sign Chart Method.


In the (1 - x) sign chart, we asked the question, when is 1 - x > 0. The answer is when 1 > x, or x < 1. This is depicted in blue.

Presentation of Solution: Solve x(1-x) < 0.

Set Notation: $\{x \mid x < 0 \text{ or } x > 1\}$ Interval Notation: $(-\infty, 0) \cup (1, +\infty)$

Here, we exclude the endpoints, 0 and 1, because they do not satisfy the stated inequality. (These points make x(1-x) equal zero.)

Example 7.9.

7.10. Solution: Begin by completely factoring the expression.

$$\frac{(x-2)(x+2)}{(x-3)} < 0 \tag{S-4}$$

We now do a **Sign Chart on** the left-hand side of (S-4).



The solution set is all real numbers that are in red.

Presentation of Solution:

$$(-\infty, -2) \cup (2, 3)$$

Example 7.10. \blacksquare

7.11. Solution: (a) Solve for x: |x - 4| < 3. $|x-4| < 3 \triangleleft \text{given}$ -3 < x - 4 < 3 \triangleleft from (16) $1 < x < 7 \triangleleft$ add 4 to all sides Presentation of Solution: (1,7)(b) Solve for x: |3x - 1| < 2. $|3x-1| < 2 \triangleleft \text{given}$ $-2 < 3x - 1 < 2 \triangleleft \text{ from (16)}$

 $-1 < 3x < 3 \quad \triangleleft \text{ add 1 to all sides}$

 $-\frac{1}{3} < x < 1$ \triangleleft multiply all sides by 1/3

Presentation of Solution: $\left(-\frac{1}{3},1\right)$

(c) Solve for
$$x: |2 - 4x| \le 5$$
.
 $|2 - 4x| \le 5 \quad \triangleleft \text{ given}$
 $-5 \le 2 - 4x \le 5 \quad \triangleleft \text{ from (16)}$
 $-7 \le -4x \le 3 \quad \triangleleft \text{ add } -2 \text{ to all sides}$
 $\frac{7}{4} \ge x \ge -\frac{3}{4} \quad \triangleleft \text{ multiply all sides by } -1/4$
or

$$-\tfrac{3}{4} \le x \le \tfrac{7}{4}$$

In the last step we have multiplied both sides by a negative number, this will *reverse the direction of the inequality*!

Presentation of Solution:
$$\left[-\frac{3}{4}, \frac{7}{4}\right]$$

Example 7.11.

7.12. Solution to (a): Solve for x: |x - 3| > 4.

$$|x - 3| > 4$$

Use (17) to **split** the inequality!

x - 3 > 4	\triangleleft upper inequality	x - 3 < -4	\triangleleft lower inequality
x > 7	\triangleleft add 3 both sides	x < -1	\triangleleft add 3 both sides
$(7, +\infty)$	\triangleleft solution set	$(-\infty, -1)$	\triangleleft solution set

Now, **join** the solutions!

Solution Set $= (7, +\infty) \cup (-\infty, -1)$

Presentation of Solution: $(-\infty, -1) \cup (7, +\infty)$

Comments That's way it goes: split, solve each, and join by union!

Solution to (b): Solve for $x: |5x + 1| \ge 3$.

 $|5x+1| \ge 3$

Use (17) to **split** the inequality!

Now, join the solutions!

Solution Set
$$= [\frac{2}{5}, +\infty) \cup (-\infty, -\frac{4}{5}]$$

Presentation of Solution: $(-\infty, -\frac{4}{5}] \cup [\frac{2}{5}, +\infty)$ Example 7.12.

Important Points

Proof: The Quadratic Formula. The quadratic formula is nothing more than the formula you get when you complete the square and solve for x just as we did in the section on completing the square. If you've been reading along this tutorial, the following steps would look familiar.

$$ax^{2} + bx + c = 0 \qquad \triangleleft \text{ given}$$

$$ax^{2} + bx = -c \qquad \triangleleft \text{ add } -c \text{ to both sides}$$

$$a(x^{2} + \frac{b}{a}x) = -c \qquad \triangleleft \text{ factor out } a \text{ from l.h.s.}$$

$$a(x^{2} + \frac{b}{a}x + \frac{b^{2}}{4a^{2}}) = \frac{b^{2}}{4a} - c \qquad \triangleleft \text{ add } b^{2}/4a \text{ to both sides}$$

$$a(x + \frac{b}{2a})^{2} = \frac{b^{2} - 4ac}{4a} \qquad \triangleleft \text{ perfect square}$$

$$(x + \frac{b}{2a})^{2} = \frac{b^{2} - 4ac}{4a^{2}} \qquad \triangleleft \text{ divide by } a$$

All the steps above are reversible. This means that the solution set to the last equation is the same as the solution set to the first equation.

You can see now that if $b^2 - 4ac < 0$, the right-hand side of the last equation is negative and the left-hand side is not; therefore, there can be no solutions to the equation.

Now let's continue the development.

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad < \text{take square root of both sides}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad < \text{extract root in denominator}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad < \text{subtract } b/2a \text{ from both sides}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad < \text{combine fractions} \qquad \text{(I-1)}$$

Now if $b^2 - 4ac = 0$, then equation (I-1) becomes $x = -\frac{b}{2a}$; that is, there is only one solution as asserted.

Finally, if $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac} > 0$ and formula (I-1) would clearly lead to two distinct solutions. Important Point

The solution to $1 \le x \ge -1$? The answer is $[1, +\infty)$. Sometimes you see students write inequalities this way, but this is not a good way of writing a double inequality!

The question is, what does $1 \le x \ge -1$ mean? It means

 $1 \le x$ and $x \ge -1$

Turn around the inequalities to get

$$x \ge 1$$
 and $x \ge -1$

Now we ask ourselves the question: What are all the numbers x that are greater than (or equal to) 1 **and** greater than (or equal to) -1? But any x satisfying $x \ge 1$ automatically satisfies $x \ge -1$; therefore, the important condition is $x \ge 1$, hence the solution set in interval notation is $[1, +\infty)$

Students who write inequalities like $1 \le x \ge -1$ often mean $x \ge 1$ or $x \ge -1$ and they just meld the two sets of inequalities together. However, the double inequality is for pairs of inequalities connected

logically by an "and". This is different from pairs of inequalities connected by a logical "or".

or

I often see students write this kind of inequality. The answer is the **empty solution:** \emptyset . Recall the meaning of double inequalities:

$$-1 \ge x \ge 1$$
 means $-1 \ge x$ and $x \ge 1$
 $x \le -1$ and $x \ge 1$

Now what number, x, is there that is less than or equal to -1 and greater than or equal to 1? The Answer: There is no such number that satisfies **both** inequalities.

What the student actually means is $x \leq -1$ or $x \geq 1$, but they do not write it correctly.

Learn to use the notation to express precisely what you mean.

Important Point

Again, the answer is **no solution:** \emptyset . If you went through the (routine) steps of solving, you would have arrived at $2 \le x \le 1$. What number x satisfies $x \ge 2$ and $x \le 1$? Answer: No number.

However, you need not have bothered to solve an inequality having the empty solution set. It is immediate from the original inequality:

$$3 \le 2x - 1 \le 1$$

Look at the left-most and right-most expressions. Do you see that

 $3 \leq 1?$

This implies immediately that there are no solutions.

Important Point